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Nonlinear superposition formulae based on the Lie group $SO(n+1,n)$

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Abstract
Systems of nonlinear ordinary differential equations are constructed, for which the general solution is algebraically expressed in terms of a finite number of particular solutions. Expressions of that type are called nonlinear superposition formulae. These systems are connected with local Lie group transformations on their homogeneous spaces. In the work presented here, the nonlinear superposition formulae are constructed for the case of the $SO(3,2)$ group and some aspects of the general case of $SO(n+1,n)$ are studied.

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1. Introduction

Nonlinear phenomena are taking on more and more importance in almost all branches of science—and especially in physics. However, in general, systems of nonlinear differential equations describing these phenomena are not practically solvable. Among the most important, at least partially solvable systems of nonlinear equations are those for which a general solution can be obtained from a finite set of particular known solutions—in other words, for which the superposition formulae are valid. These systems of nonlinear equations are connected with Lie groups and their action on homogeneous spaces. The best-known example is the Riccati equation which is connected with the Lie group $SL(2)$. Equations of this type were classified in [1] and explicitly found for $SL(n)$ in [2], and for the complex symplectic group in [3]. Another problem is that of finding the corresponding superposition formulae for these equations. These problems have been solved for the particular cases of equations connected

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with the Lie groups $SL(n)$, in [4–8], $SP(2n)$, in [4], and $SO(n,n)$, in [9]—that is, for the Lie groups of the type A_n, C_n and D_n. Our aim is to find the superposition formulae for the Lie group B_n, and in particular for B_2. It was shown in [1] that the nonlinear structures of the systems of equations connected with the Lie groups $SO(p,q)$ are usually polynomials of degree 4. The exception is $SO(n+1,n)$. In this case, the nonlinear structures are polynomials of degree 2. In this paper, we restrict consideration to this simple case.

In general, the systems of nonlinear equations connected with the Lie groups, including pseudo-orthogonal groups, occur in many physical applications such as in Bäcklund transformations in the study of integrable systems, as special cases of Volterra–Lotke equations in population dynamics, in optimal control theory, in chaos, in symplectic optics and elsewhere [10–15]. Therefore, the superposition formulae are interesting from both the mathematical and physical points of view.

Let us start with a brief formulation of our problem. Consider a system of n first-order differential equations:

$$\dot{x}^\mu(t) = \chi^\mu(x^1, x^2, \ldots, x^n, t) \qquad \mu = 1, \ldots, n \tag{1}$$

where the dot denotes differentiation of $x^\mu(t)$ with respect to time t. It has been known for a very long time that in some cases, which we will be specified later, it is possible to express the general solution as a nonlinear function of a finite number of particular solutions; it is of the form

$$\boldsymbol{x} = \boldsymbol{F}(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m, c_1, \ldots, c_s) \qquad \boldsymbol{x} \in \boldsymbol{R}^n \tag{2}$$

where $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$ are the particular solutions (1), $c_1, c_2, \ldots, c_s$ are arbitrary constants and $\boldsymbol{x}(t)$ is the general solution. These relations are called nonlinear superposition formulae. Here $\boldsymbol{x}$ means a vector with elements $x^1, x^2, \ldots, x^n$.

An example of such systems is a homogeneous system of the first-order linear differential equations, in which the general solution is expressed as a linear combination of n linearly independent particular solutions. The other known example is the Riccati equation

$$\dot{x} = a(t) + b(t)x + c(t)x^2 \tag{3}$$

where $a(t)$, $b(t)$, $c(t)$ are continuous differentiable functions with respect to t. In this case, for any four solutions $x_i(t)$, $i = 1, \ldots, 4$, the relation

$$\frac{x_1(t) - x_3(t)}{x_1(t) - x_4(t)} \frac{x_2(t) - x_4(t)}{x_2(t) - x_3(t)} = \frac{u_1 - u_3}{u_1 - u_4} \frac{u_2 - u_4}{u_2 - u_3} \tag{4}$$

where $x_i(0) = u_i$ are initial conditions, is valid.

In the general case, such systems of differential equations are connected with the local Lie group G of transformations on the factor space $M = G/G_0$, where G_0 is a Lie subalgebra of G [16]. We recall this connection briefly.

By the local Lie group G of transformations on M, we understand a smooth mapping $\varphi : G \times M \to M$ (we use the abbreviation $\varphi(g, u) = g \cdot u$), for which

(a) $e \cdot u = u$, for any $u \in M$, where e is the unit element of the group G,
(b) for any two elements $g_1, g_2 \in G$ and any $u \in M$, we have $g_2 \cdot (g_1 \cdot u) = (g_2 g_1) \cdot u$ and
(c) $g \cdot u = u$ for any $u \in M$ implies $g = e$ [17].

In the local coordinate system, we write $x = g \cdot u$ as

$$x^\mu = f^\mu(a^1, \ldots, a^N, u^1, \ldots, u^n) \qquad \mu = 1, \ldots, n \tag{5}$$

where N is the dimension of the group G and a^r, $r = 1, \ldots, N$, are their local coordinates. For $x^\mu(\boldsymbol{a}, \boldsymbol{u})$, we can write

$$\frac{\partial x^\mu}{\partial a^r} = \sum_{s=1}^{N} \xi_s^\mu(\boldsymbol{x}) v_r^s(\boldsymbol{a}) \tag{6}$$

where the vector fields

$$X_s(\boldsymbol{x}) = -\sum_{\mu=1}^{n} \xi_s^\mu(\boldsymbol{x}) \frac{\partial}{\partial x^\mu}$$

obey the equations

$$[X_r, X_s] = \sum_{t=1}^{N} c_{rs}^t X_t$$

in which c_{rs}^t are the structure constants of the Lie algebra of the group G.

Conversely, the vector fields $X_k(\boldsymbol{x})$ uniquely determine the infinitesimal action of the local Lie group on the space M [17].

Let $g(t) = (a^1(t), \dots, a^N(t))$, $t \in \boldsymbol{R}$, be a curve in the Lie group G such that $g(0) = e$. This gives a curve in the space M. Differentiation of equation (5) with respect to the parameter t gives, using (6), the system of differential equations

$$\dot{x}^\mu = \sum_{r=1}^{N} \xi_r^\mu(\boldsymbol{x}) Z^r(t) \qquad \mu = 1, \dots, n. \tag{7}$$

In this paper we will deal with systems of this kind, connected with the Lie group $SO(n+1, n)$.

If the system of equations (1) has the form (7), then there is a curve in some local coordinates of the Lie group G, which acts on the factor space M. In this case, it is possible to find the superposition formula [16]. Any particular solution of the system (7) can be written in the form

$$\boldsymbol{x}_k(t) = g(t) \cdot \boldsymbol{u}_k \tag{8}$$

where $\boldsymbol{u}_k = \boldsymbol{x}_k(0)$ is the initial condition.

We express the action of the local group G by using the action of this group on a few points of the space M, which is assumed known. In principle, this means finding the coordinates of the group a^i from the system of equations

$$\begin{aligned} \boldsymbol{x}_1 &= \boldsymbol{f}(a^1, \dots, a^N, \boldsymbol{u}_1) \\ \boldsymbol{x}_2 &= \boldsymbol{f}(a^1, \dots, a^N, \boldsymbol{u}_2) \\ &\dots \\ \boldsymbol{x}_r &= \boldsymbol{f}(a^1, \dots, a^N, \boldsymbol{u}_r). \end{aligned} \tag{9}$$

To find the group coordinates a^i, $i = 1, \dots, N$, we should use the action of r points. It is evident that the number r must fulfil the inequality $nr \geqslant N$, where n is the dimension of M and N is the dimension of the group G.

If we solve this problem, then we are able to express the elements of the Lie group G by means of the known transformations of the r points in the form

$$g = g(\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_r, \boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_r); \tag{10}$$

therefore,

$$\boldsymbol{x} = g(\boldsymbol{x}_1, \dots, \boldsymbol{x}_r, \boldsymbol{u}_1, \dots, \boldsymbol{u}_r) \cdot \boldsymbol{u} \tag{11}$$

holds. Now we see that formula (11) is invariant with respect to the action of the local group G.

If $\boldsymbol{x}_i(t)$, $i = 1, \dots, r$, are the known solutions of the system (7) for given functions $Z^r(t)$, then any other solution $\boldsymbol{x}(t)$ of that system is given by (11). Therefore, relation (11) is the superposition formula [16].

For example, the Riccati equation (3) is connected with the Lie group $G = SL(2)$ that acts on the space M as follows:

$$g(t) \cdot u = x(t) = \frac{a_{21}(t) + ua_{22}(t)}{a_{11}(t) + ua_{12}(t)}$$

where we represented the elements of the group G by the matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

with the determinant equal to 1.

In the 1980s, there were a lot of papers [4–9] which were devoted to systems of differential equations of this type and especially to finding the superposition formulae for these systems. For the most part, the authors studied the systems connected with Lie groups $SL(n, \boldsymbol{R})$ or $SL(n, \boldsymbol{C})$. In the general paper [16], techniques used for constructing the superposition formulae were described. They are demonstrated using examples, in which the Lie groups $SL(n, \boldsymbol{R})$ and $O(p + 1, n - p + 1)$ act on simple projective spaces.

In [4], the more general cases of the projective-matrix Riccati equation are studied for $SL(2n, \boldsymbol{R})$ and $SP(2n, \boldsymbol{R})$. In paper [5], the systems that arise from the action of $SL(n, \boldsymbol{C})$ on the factor spaces $SL(n, \boldsymbol{C})/O(n, \boldsymbol{C})$ and $SL(2n, \boldsymbol{C})/SP(2n, \boldsymbol{C})$ are studied. The paper [6] is devoted to the superposition formulae for the rectangular-matrix Riccati equations on the space $SL(n + k, \boldsymbol{C})/P(k)$, where $P(k)$ are special maximal parabolic subgroups of $SL(n + k, \boldsymbol{C})$. In [7], the authors deal with systems connected with the Lie group $SU(n, n)$ and, in [9], the same method is used for $SO(n, n)$. Finally, in the paper [8], the authors study the systems of equations that are connected with the action of the Lie group $SL(n, \boldsymbol{C})$ on the space $M = SL(n, \boldsymbol{C})/G_0$, where G_0 is a special non-maximal parabolic subgroup.

The authors mostly used a set of special solutions for reconstructing the group action on the space M. This approach simplifies the solution of (9). On the other hand, we are not able to use the resulting superposition formulae directly for constructing the solution of the system on the basis of any set of particular solutions. By means of the action of group elements, we must first transform our particular solutions to a special set of particular solutions used in superposition formulae.

In the next section, we study the systems of equations (7) which are connected with the action of the Lie group $SO(n+1, n)$ on the space $M = SO(n+1, n)/P$, where P is one of the maximal parabolic subgroups. To our knowledge, systems of this type have not been studied so far. Unlike the authors of the papers cited above, we do not choose a special set of particular solutions for reconstructing the group action on M, or, in the terminology of paper [16], we construct group invariants which give nonlinear superposition formulae in implicit form.

2. Lie group $SO(n + 1, n)$ and its Lie algebra

In this section, we set up the notation. The Lie group $SO(n + 1, n)$ is a group of real matrices $\boldsymbol{G}$ with dimension $(2n + 1) \times (2n + 1)$ that fulfil the equations

$$\boldsymbol{G}^{\mathrm{T}} \cdot \sigma \cdot \boldsymbol{G} = \sigma \qquad \text{where } \sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \boldsymbol{I} \\ 0 & \boldsymbol{I} & 0 \end{pmatrix} \tag{12}$$

$\boldsymbol{G}^{\mathrm{T}}$ denotes a transposed matrix and $\boldsymbol{I}$ is a unit matrix with dimension $n \times n$.

Matrix $\boldsymbol{G}$ is written in the form

$$\boldsymbol{G} = \begin{pmatrix} g_{11} & \boldsymbol{g}_{12}^{\mathrm{T}} & \boldsymbol{g}_{13}^{\mathrm{T}} \\ \boldsymbol{g}_{21} & \boldsymbol{G}_{22} & \boldsymbol{G}_{23} \\ \boldsymbol{g}_{31} & \boldsymbol{G}_{32} & \boldsymbol{G}_{33} \end{pmatrix} \tag{13}$$

where $\boldsymbol{g}$ denotes a column vector and $\boldsymbol{g}^{\mathrm{T}}$ is its transpose represented as row vector. Therefore $\boldsymbol{g}\boldsymbol{g}^{\mathrm{T}}$ is an $n \times n$ matrix and $\boldsymbol{g}^{\mathrm{T}}\boldsymbol{g}$ is an inner product.

If we insert (13) into (12) we obtain the following formulae for elements of the matrix $\boldsymbol{G}$:

$$\begin{aligned}
&g_{11}^2 + \boldsymbol{g}_{21}^{\mathrm{T}}\boldsymbol{g}_{31} + \boldsymbol{g}_{31}^{\mathrm{T}}\boldsymbol{g}_{21} = 1\\
&g_{11}\boldsymbol{g}_{12} + \boldsymbol{G}_{22}^{\mathrm{T}}\boldsymbol{g}_{31} + \boldsymbol{G}_{32}^{\mathrm{T}}\boldsymbol{g}_{21} = 0\\
&g_{11}\boldsymbol{g}_{13} + \boldsymbol{G}_{23}^{\mathrm{T}}\boldsymbol{g}_{31} + \boldsymbol{G}_{33}^{\mathrm{T}}\boldsymbol{g}_{21} = 0\\
&\boldsymbol{g}_{12}\boldsymbol{g}_{12}^{\mathrm{T}} + \boldsymbol{G}_{22}^{\mathrm{T}}\boldsymbol{G}_{32} + \boldsymbol{G}_{32}^{\mathrm{T}}\boldsymbol{G}_{22} = 0\\
&\boldsymbol{g}_{13}\boldsymbol{g}_{13}^{\mathrm{T}} + \boldsymbol{G}_{23}^{\mathrm{T}}\boldsymbol{G}_{33} + \boldsymbol{G}_{33}^{\mathrm{T}}\boldsymbol{G}_{23} = 0\\
&\boldsymbol{g}_{12}\boldsymbol{g}_{13}^{\mathrm{T}} + \boldsymbol{G}_{22}^{\mathrm{T}}\boldsymbol{G}_{33} + \boldsymbol{G}_{32}^{\mathrm{T}}\boldsymbol{G}_{23} = \boldsymbol{I}.
\end{aligned} \tag{14}$$

In this realization, the Lie algebra $so(n+1, n)$ is given by real matrices:

$$\boldsymbol{A} = \begin{pmatrix} 0 & \boldsymbol{x}^{\mathrm{T}} & \boldsymbol{z}^{\mathrm{T}} \\ -\boldsymbol{z} & \boldsymbol{H} & \boldsymbol{W} \\ -\boldsymbol{x} & \boldsymbol{Y} & -\boldsymbol{H}^{\mathrm{T}} \end{pmatrix}$$

where $\boldsymbol{W}^{\mathrm{T}} = -\boldsymbol{W}$ and $\boldsymbol{Y}^{\mathrm{T}} = -\boldsymbol{Y}$.

In the group $SO(n+1, n)$, we take a subgroup G_0 that is generated by the matrices

$$\boldsymbol{G}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \boldsymbol{D} & 0 \\ 0 & 0 & (\boldsymbol{D}^{\mathrm{T}})^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & \boldsymbol{z}^{\mathrm{T}} \\ -\boldsymbol{z} & \boldsymbol{I} & \boldsymbol{Z} \\ 0 & 0 & \boldsymbol{I} \end{pmatrix}$$

where the equality $\boldsymbol{Z} + \boldsymbol{Z}^{\mathrm{T}} + \boldsymbol{z}\boldsymbol{z}^{\mathrm{T}} = 0$ holds. The factor space $M = G/G_0$ can be represented by the matrices

$$\Xi = \begin{pmatrix} 1 & \boldsymbol{x}^{\mathrm{T}} & 0 \\ 0 & \boldsymbol{I} & 0 \\ -\boldsymbol{x} & \boldsymbol{X} & \boldsymbol{I} \end{pmatrix}$$

where $\boldsymbol{X} + \boldsymbol{X}^{\mathrm{T}} + \boldsymbol{x}\boldsymbol{x}^{\mathrm{T}} = 0$. As coordinates on M, we choose $\boldsymbol{x}$ and the antisymmetric part of the matrix $\boldsymbol{X}$; this means that

$$\boldsymbol{Y} = \boldsymbol{X} + \tfrac{1}{2}\boldsymbol{x}\boldsymbol{x}^{\mathrm{T}}. \tag{15}$$

The action of the group G on the factor space M in these coordinates can be obtained from the equation

$$\begin{aligned}
&\begin{pmatrix} g_{11} & \boldsymbol{g}_{12}^{\mathrm{T}} & \boldsymbol{g}_{13}^{\mathrm{T}} \\ \boldsymbol{g}_{21} & \boldsymbol{G}_{22} & \boldsymbol{G}_{23} \\ \boldsymbol{g}_{31} & \boldsymbol{G}_{32} & \boldsymbol{G}_{33} \end{pmatrix} \begin{pmatrix} 1 & \boldsymbol{u}^{\mathrm{T}} & 0 \\ 0 & \boldsymbol{I} & 0 \\ -\boldsymbol{u} & \boldsymbol{U} & \boldsymbol{I} \end{pmatrix}\\
&\qquad = \begin{pmatrix} 1 & \boldsymbol{x}^{\mathrm{T}} & 0 \\ 0 & \boldsymbol{I} & 0 \\ -\boldsymbol{x} & \boldsymbol{X} & \boldsymbol{I} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \boldsymbol{D} & 0 \\ 0 & 0 & (\boldsymbol{D}^{\mathrm{T}})^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & \boldsymbol{z}^{\mathrm{T}} \\ -\boldsymbol{z} & \boldsymbol{I} & \boldsymbol{Z} \\ 0 & 0 & \boldsymbol{I} \end{pmatrix}.
\end{aligned}$$

If we compare the coefficients on either side of this equation, we obtain the following formulae:

$$\begin{aligned}
\boldsymbol{D} &= \boldsymbol{G}_{22} + \boldsymbol{g}_{21}\boldsymbol{u}^{\mathrm{T}} + \boldsymbol{G}_{23}\boldsymbol{U}\\
\boldsymbol{x} &= (\boldsymbol{D}^{\mathrm{T}})^{-1}(\boldsymbol{g}_{12} + g_{11}\boldsymbol{u} + \boldsymbol{U}^{\mathrm{T}}\boldsymbol{g}_{13})\\
\boldsymbol{X} &= (\boldsymbol{G}_{32} + \boldsymbol{g}_{31}\boldsymbol{u}^{\mathrm{T}} + \boldsymbol{G}_{33}\boldsymbol{U})\boldsymbol{D}^{-1}
\end{aligned} \tag{16}$$

for the group elements. Because we restrict consideration to the local Lie group, we can suppose that matrix $\boldsymbol{D}$ is invertible.

Starting with the action of this group on the space M and using the expansion to first order, we derive an explicit expression for the vector fields, in the basis of the algebra $so(n+1, n)$ in the representation $(T_g f)(m) = f(g^{-1} \cdot m)$. Specifically we get

$$
\begin{aligned}
Y_{ij} &\mapsto -\frac{\partial}{\partial Y_{ij}} \\
x_i &\mapsto -\frac{\partial}{\partial x_i} - \frac{1}{2}\sum_{r=1}^{n} x_r \frac{\partial}{\partial Y_{ri}} \\
D_{ij} &\mapsto x_j \frac{\partial}{\partial x_i} - \sum_{r=1}^{n} Y_{ir} \frac{\partial}{\partial Y_{rj}} \\
z_i &\mapsto -\sum_{r=1}^{n}\left(Y_{ir} + \frac{x_i x_r}{2}\right)\frac{\partial}{\partial x_r} - \frac{1}{2}\sum_{r,s=1}^{n} Y_{ri} x_s \frac{\partial}{\partial Y_{rs}} \\
W_{ij} &\mapsto \sum_{r=1}^{n}(x_j Y_{ri} - x_i Y_{rj})\frac{\partial}{\partial x_r} - \sum_{r,s=1}^{n} Y_{ir} Y_{js} \frac{\partial}{\partial Y_{rs}}
\end{aligned}
$$

where we define $Y_{rs} = -Y_{sr}$.

The system of differential equations (7) is in this case of the form

$$
\begin{aligned}
\dot{\boldsymbol{x}}(t) &= \boldsymbol{a}(t) - \boldsymbol{B}(t)\boldsymbol{x} - \boldsymbol{Y}\boldsymbol{c}(t) + \tfrac{1}{2}\boldsymbol{x}\boldsymbol{c}^{\mathrm{T}}(t)\boldsymbol{x} - \boldsymbol{Y}\boldsymbol{C}(t)\boldsymbol{x} \\
\dot{\boldsymbol{Y}}(t) &= \boldsymbol{A}(t) + \tfrac{1}{2}(\boldsymbol{x}\boldsymbol{a}^{\mathrm{T}}(t) - \boldsymbol{a}(t)\boldsymbol{x}^{\mathrm{T}}) - (\boldsymbol{Y}\boldsymbol{B}(t) + \boldsymbol{B}^{\mathrm{T}}(t)\boldsymbol{Y}) \\
&\quad + \tfrac{1}{2}(\boldsymbol{Y}\boldsymbol{c}(t)\boldsymbol{x}^{\mathrm{T}} + \boldsymbol{x}\boldsymbol{c}^{\mathrm{T}}(t)\boldsymbol{Y}) - \boldsymbol{Y}\boldsymbol{C}(t)\boldsymbol{Y}
\end{aligned}
\tag{17}
$$

where $\boldsymbol{a}(t)$ and $\boldsymbol{c}(t)$ are vector functions differentiable with respect to t; $\boldsymbol{A}(t)$, $\boldsymbol{B}(t)$ and $\boldsymbol{C}(t)$ are differentiable matrix functions, for which $\boldsymbol{A}^{\mathrm{T}} = -\boldsymbol{A}$, $\boldsymbol{C}^{\mathrm{T}} = -\boldsymbol{C}$ and $\boldsymbol{Y}^{\mathrm{T}} = -\boldsymbol{Y}$.

This system of differential equation is the same as in the paper [1], where a classification of all systems of nonlinear ordinary differential equations of this type is given.

3. Representations of the action of the group by means of particular solutions

The system of equations (17) arises from the action of the Lie group $SO(n+1, n)$ on the factor space $M = SO(n+1, n)/G_0$. Therefore, one can find the superposition formula. The action of the group $SO(n+1, n)$ on the space M is given by the relations (16). We try to express this action by means of any known solutions $\boldsymbol{x}_k = g \cdot \boldsymbol{u}_k$. If $\boldsymbol{x}_k(t)$ are solutions of the differential equations (17) with the initial conditions $\boldsymbol{x}_k(0) = \boldsymbol{u}_k$, we obtain the general solution $\boldsymbol{x}(t)$ with the initial condition $\boldsymbol{x}(0) = \boldsymbol{u}$ in the form

$$
\boldsymbol{x}(t) = G(\boldsymbol{x}_1(t), \ldots, \boldsymbol{x}_r(t), \boldsymbol{u}_1, \ldots, \boldsymbol{u}_r) \cdot \boldsymbol{u}
$$

where $g(\boldsymbol{x}_1(t), \ldots, \boldsymbol{x}_r(t), \boldsymbol{u}_1, \ldots, \boldsymbol{u}_r) = g(t)$ is the expression for an element of the group G in terms of the known transformation.

The dimension of the group $SO(n+1, n)$ is equal to $n(2n+1)$ and the dimension of the space $M = SO(n+1, n)/G_0$ is $n(n+1)/2$. So, to express the N coordinates of the group elements by using the known solutions, we must know at least r solutions, where r fulfils the inequality $n(2n+1) \leqslant [n(n+1)/2]r$. Consequently, for $n = 1$, it is sufficient to known three particular solutions, and for $n > 1$, we must know at least four particular solutions of the system (17).

By $\boldsymbol{x}_i$ we denote a vector, and by $\boldsymbol{X}_i = \boldsymbol{Y}_i - \frac{1}{2}\boldsymbol{x}_i\boldsymbol{x}_i^{\mathrm{T}}$ we denote a matrix for and ith solution of the system of differential equations (17) with initial conditions $\boldsymbol{u}_i$ and $\boldsymbol{U}_i$; $\boldsymbol{D}_i$ is an invertible matrix defined to provide an appropriate solution of the first equation in (16). Further, we use the notation $\boldsymbol{u}_{ik} = \boldsymbol{u}_i - \boldsymbol{u}_k$, $\boldsymbol{x}_{ik} = \boldsymbol{x}_i - \boldsymbol{x}_k$. Similarly, we define $\boldsymbol{X}_{ik}$ and $\boldsymbol{U}_{ik}$. Temporarily, we suppose that all matrices which we will use in our calculations are invertible.

From equation (16) we obtain some coordinates of elements of the group G. Readily we discover that for indices i and k, the relations

$$\begin{aligned}
\boldsymbol{g}_{13} &= (\boldsymbol{U}_{ik}^{\mathrm{T}})^{-1} \cdot (\boldsymbol{D}_i^{\mathrm{T}} \boldsymbol{x}_i - \boldsymbol{D}_k^{\mathrm{T}} \boldsymbol{x}_k - g_{11} \boldsymbol{u}_{ik}) \\
\boldsymbol{G}_{23} &= (\boldsymbol{D}_i - \boldsymbol{D}_k - \boldsymbol{g}_{21} \boldsymbol{u}_{ik}^{\mathrm{T}}) \cdot \boldsymbol{U}_{ik}^{-1} \\
\boldsymbol{G}_{33} &= (\boldsymbol{X}_i \boldsymbol{D}_i - \boldsymbol{X}_k \boldsymbol{D}_k - \boldsymbol{g}_{31} \boldsymbol{u}_{ik}^{\mathrm{T}}) \cdot \boldsymbol{U}_{ik}^{-1} \\
\boldsymbol{g}_{12} &= \boldsymbol{D}_i^{\mathrm{T}} \boldsymbol{x}_i - g_{11} \boldsymbol{u}_i - \boldsymbol{U}_i^{\mathrm{T}} \boldsymbol{g}_{13} \\
\boldsymbol{G}_{22} &= \boldsymbol{D}_i - \boldsymbol{g}_{21} \boldsymbol{u}_i^{\mathrm{T}} - \boldsymbol{G}_{23} \boldsymbol{U}_i \\
\boldsymbol{G}_{32} &= \boldsymbol{X}_i \boldsymbol{D}_i - \boldsymbol{g}_{31} \boldsymbol{u}_i^{\mathrm{T}} - \boldsymbol{G}_{33} \boldsymbol{U}_i
\end{aligned}$$

are valid. If we now apply these equations to (14), after some simple algebra we obtain the following formulae:

$$\begin{aligned}
&g_{11}^2 + \boldsymbol{g}_{21}^{\mathrm{T}} \boldsymbol{g}_{31} + \boldsymbol{g}_{31}^{\mathrm{T}} \boldsymbol{g}_{21} = 1 \\
&\boldsymbol{D}_i^{\mathrm{T}} \cdot (g_{11} \boldsymbol{x}_i + \boldsymbol{g}_{31} + \boldsymbol{X}_i^{\mathrm{T}} \boldsymbol{g}_{21}) = \boldsymbol{u}_i \\
&\boldsymbol{D}_i^{\mathrm{T}} \cdot (\boldsymbol{x}_i \boldsymbol{x}_k^{\mathrm{T}} + \boldsymbol{X}_i^{\mathrm{T}} + \boldsymbol{X}_k) \cdot \boldsymbol{D}_k = \boldsymbol{u}_i \boldsymbol{u}_k^{\mathrm{T}} + \boldsymbol{U}_i^{\mathrm{T}} + \boldsymbol{U}_k.
\end{aligned} \tag{18}$$

Using the second equation in (18) we can write

$$\begin{aligned}
\boldsymbol{g}_{21} &= -(\boldsymbol{X}_{ik}^{\mathrm{T}})^{-1} [g_{11} \boldsymbol{x}_i - (\boldsymbol{D}_i^{\mathrm{T}})^{-1} \boldsymbol{u}_i] - (\boldsymbol{X}_{ki}^{\mathrm{T}})^{-1} [g_{11} \boldsymbol{x}_k - (\boldsymbol{D}_k^{\mathrm{T}})^{-1} \boldsymbol{u}_k] \\
\boldsymbol{g}_{31} &= \boldsymbol{X}_k^{\mathrm{T}} (\boldsymbol{X}_{ik}^{\mathrm{T}})^{-1} [g_{11} \boldsymbol{x}_i - (\boldsymbol{D}_i^{\mathrm{T}})^{-1} \boldsymbol{u}_i] + \boldsymbol{X}_i^{\mathrm{T}} (\boldsymbol{X}_{ki}^{\mathrm{T}})^{-1} [g_{11} \boldsymbol{x}_k - (\boldsymbol{D}_k^{\mathrm{T}})^{-1} \boldsymbol{u}_k].
\end{aligned} \tag{19}$$

Next we use the notation

$$\begin{aligned}
\Omega_{ik} &= \boldsymbol{x}_i \boldsymbol{x}_k^{\mathrm{T}} + \boldsymbol{X}_i^{\mathrm{T}} + \boldsymbol{X}_k \\
\omega_{ik} &= \boldsymbol{u}_i \boldsymbol{u}_k^{\mathrm{T}} + \boldsymbol{U}_i^{\mathrm{T}} + \boldsymbol{U}_k \\
\boldsymbol{h}_i &= g_{11} \boldsymbol{x}_i - (\boldsymbol{D}_i^{\mathrm{T}})^{-1} \boldsymbol{u}_i.
\end{aligned} \tag{20}$$

It is easy to see that the relations

$$\begin{aligned}
&\boldsymbol{x}_i \boldsymbol{x}_k^{\mathrm{T}} + \Omega_{ik} + \Omega_{ki} = 0 \qquad && \Omega_{ik}^{\mathrm{T}} = \Omega_{ki} \\
&\boldsymbol{u}_i \boldsymbol{u}_k^{\mathrm{T}} + \omega_{ik} + \omega_{ki} = 0 \qquad && \omega_{ik}^{\mathrm{T}} = \omega_{ki}
\end{aligned}$$

are valid. By simple algebraic calculations, from equations (18) and (19) for any i, j and k, we derive the following formulae:

$$\boldsymbol{D}_i^{\mathrm{T}} \Omega_{ik} \boldsymbol{D}_k = \omega_{ik} \tag{21}$$

$$\boldsymbol{h}_k + \boldsymbol{X}_{jk}^{\mathrm{T}} \cdot (\boldsymbol{X}_{ij}^{\mathrm{T}})^{-1} \boldsymbol{h}_i + \boldsymbol{X}_{ik}^{\mathrm{T}} \cdot (\boldsymbol{X}_{ji}^{\mathrm{T}})^{-1} \boldsymbol{h}_j = 0 \tag{22}$$

$$\begin{aligned}
g_{11}^2 &+ [\boldsymbol{h}_k^{\mathrm{T}} \boldsymbol{X}_{ik}^{-1} \boldsymbol{x}_k + \boldsymbol{h}_k^{\mathrm{T}} \boldsymbol{X}_{ki}^{-1} \boldsymbol{x}_i] \cdot [\boldsymbol{x}_k^{\mathrm{T}} (\boldsymbol{X}_{ik}^{\mathrm{T}})^{-1} \boldsymbol{h}_i + \boldsymbol{x}_i^{\mathrm{T}} (\boldsymbol{X}_{ki}^{\mathrm{T}})^{-1} \boldsymbol{h}_k] \\
&- \boldsymbol{h}_i^{\mathrm{T}} \boldsymbol{X}_{ik}^{-1} \Omega_{ik} (\boldsymbol{X}_{ki}^{\mathrm{T}})^{-1} \boldsymbol{h}_k - \boldsymbol{h}_k^{\mathrm{T}} \boldsymbol{X}_{ki}^{-1} \Omega_{ki} (\boldsymbol{X}_{ik}^{\mathrm{T}})^{-1} \boldsymbol{h}_i = 1.
\end{aligned} \tag{23}$$

We have found the system of equations from which it is possible to determine the matrix (13). Since we restricted consideration to the neighbourhood of the point $t = 0$, we can suppose that matrices $\boldsymbol{D}_i(t)$ are invertible. From the relation (21) it follows that in this neighbourhood the matrices $\Omega_{ik}(k)$ and ω_{ki} are simultaneously invertible or non-invertible. So, it is easy to see that, if matrix ω_{ik} is invertible, it is enough to know only one solution of this system, $\boldsymbol{D}_i$, for one value of i. We can find the other $\boldsymbol{D}_k$ from equation (21). If we know $\boldsymbol{D}_i$ explicitly, equation (23) is a quadratic equation with respect to g_{11}. The odd equations can then give the matrices $\boldsymbol{D}_i$.

We summarize the above in terms of the following theorem.

Theorem. *Let* $\boldsymbol{x}_i(t)$ *and* $\boldsymbol{Y}_i(t)$, $i = 1, 2, 3$, *be three solutions of equation (17),* $\boldsymbol{X}_i(t) = \boldsymbol{Y}_i(t) - \frac{1}{2}\boldsymbol{x}_i \boldsymbol{x}_i^{\mathrm{T}}$, $\boldsymbol{u}_i = \boldsymbol{x}_i(0)$, $\boldsymbol{U}_i = \boldsymbol{X}_i(0)$ *and let the matrices* $\boldsymbol{U}_{ik}$ *be invertible. Then, there* *are a neighbourhood of the point* $t = 0$, *matrices* $\boldsymbol{D}_i(t)$, $i = 1, 2, 3$, $\boldsymbol{D}_i(0) = \boldsymbol{I}$, *and a function* $g_{11}(t)$, $g_{11}(0) = 1$, *for which formulae (21), (22), and (23) are true.*

4. The superposition formulae

In the previous sections, we formulated conditions (21)–(23) that are valid for any three solutions of the system (17). Although we are not able to solve the system (21)–(23) explicitly, it is possible to derive, from them, certain relations for solutions of the system of differential equations.

We suppose that we have five solutions of this system, and there exists a matrix inverse to ω_{ik} for any $i \neq k$, $i, k = 1, \ldots, 5$. Then from equation (21), we obtain

$$\boldsymbol{D}_k^{-1}\Omega_{rk}^{-1}(\boldsymbol{D}_r^{\mathrm{T}})^{-1} = \omega_{rk}^{-1}.$$

If we now multiply equation (21) by this equation from the left, and then by (21) from the left, for the couple (ri), we obtain the equation

$$\boldsymbol{D}_i^{\mathrm{T}}\Omega_{ik}\Omega_{rk}^{-1}\Omega_{ri}\boldsymbol{D}_i = \omega_{ik}\omega_{rk}^{-1}\omega_{ri} \tag{24}$$

which is true for any i, k and r. We multiply this equation further from the right by the inverse equation (24) for the triple (sti). Then, we obtain

$$\boldsymbol{D}_i^{-1}\Omega_{ti}^{-1}\Omega_{ts}\Omega_{is}^{-1}\Omega_{ik}\Omega_{rk}^{-1}\Omega_{ri}\boldsymbol{D}_i = \omega_{ti}^{-1}\omega_{ts}\omega_{is}^{-1}\omega_{ik}\omega_{rk}^{-1}\omega_{ri}$$

which is true for any $(ikrst)$. If now we put $s = k$ in this equation, we obtain, for any four solutions of (17), the relation

$$\boldsymbol{D}_i^{-1}\Omega_{si}^{-1}\Omega_{sk}\Omega_{rk}^{-1}\Omega_{ri}\boldsymbol{D}_i = \omega_{si}^{-1}\omega_{sk}\omega_{rk}^{-1}\omega_{ri}. \tag{25}$$

This means that the matrices $\Omega_{si}^{-1}\Omega_{sk}\Omega_{rk}^{-1}\Omega_{ri}$ and $\omega_{si}^{-1}\omega_{sk}\omega_{rk}^{-1}\omega_{ri}$ are similar. Therefore, all their invariants are identical.

Now, we will use this interesting property of the solution of differential equations (17) to obtain superposition formulae of the system (17) for small n.

For $n = 1$ we get the Lie group $SO(2, 1)$. In this case, the vector $\boldsymbol{x}$ is reduced to the number x, and $\boldsymbol{Y} = 0$. From this we have $\boldsymbol{X}_i = -\frac{1}{2}x_i^2$ and $\Omega_{ik} = -\frac{1}{2}(x_i - x_k)^2$. In this case, equation (25) has, after the extraction, the form

$$\frac{x_s - x_k}{x_s - x_i}\frac{x_r - x_i}{x_r - x_k} = \frac{u_s - u_k}{x_u - u_i}\frac{u_r - u_i}{u_r - u_k}.$$

This represents the well-known superposition formula (4) for the Riccati equation (3). This is a consequence of the isomorphism between $so(2, 1)$ and $sl(2)$.

Now, we will study the case $n = 2$—that is, the Lie group $SO(3, 2)$. In this case, we have

$$\boldsymbol{Y} = \begin{pmatrix} 0 & x_3 \\ -x_3 & 0 \end{pmatrix} \qquad \boldsymbol{X} = \frac{1}{2}\begin{pmatrix} -x_1^2 & -x_1x_2 + 2x_3 \\ -x_1x_2 - 2x_3 & -x_2^2 \end{pmatrix}.$$

If we denote the components of the ith solution by $x_1^{(i)}$, $x_2^{(i)}$ and $x_3^{(i)}$, we obtain

$$\det(\boldsymbol{U}_{ik}) = \tfrac{1}{4}(4(u_3^{(i)} - u_3^{(k)})^2 - (u_2^{(i)}u_1^{(k)} - u_1^{(i)}u_2^{(k)})^2)$$

and the determinants of the matrices Ω_{ik} and ω_{ik} are

$$\det(\Omega_{ik}) = \tfrac{1}{4}\Delta_{ik}^2 \qquad \det(\omega_{ik}) = \tfrac{1}{4}\delta_{ik}^2 \tag{26}$$

where

$$\Delta_{ik} = 2x_3^{(i)} - 2x_3^{(k)} + x_2^{(i)}x_1^{(k)} - x_1^{(i)}x_2^{(k)}$$
$$\delta_{ik} = 2u_3^{(i)} - 2u_3^{(k)} + u_2^{(i)}u_1^{(k)} - u_1^{(i)}u_2^{(k)}.$$

We see that the conditions for the matrices $\boldsymbol{U}_{ik}$ to be invertible imply the invertibility of matrices ω_{ik}.

If we now take the determinant in equation (25) we obtain for any four different solutions the equality

$$\frac{\Delta_{sk}}{\Delta_{si}}\frac{\Delta_{ri}}{\Delta_{rk}} = \frac{\delta_{sk}}{\delta_{si}}\frac{\delta_{ri}}{\delta_{rk}}. \tag{27}$$

As was mentioned above, in this case, we can construct the general solution of the system (17) by using four particular solutions. Take now five solutions, for which from (27) we obtain independent equations

$$\frac{\Delta_{24}}{\Delta_{14}}\frac{\Delta_{13}}{\Delta_{23}} = \frac{\delta_{24}}{\delta_{14}}\frac{\delta_{13}}{\delta_{23}} \qquad \frac{\Delta_{34}}{\Delta_{14}}\frac{\Delta_{12}}{\Delta_{23}} = \frac{\delta_{34}}{\delta_{14}}\frac{\delta_{12}}{\delta_{23}} \tag{28}$$

$$\frac{\Delta_{25}}{\Delta_{15}}\frac{\Delta_{13}}{\Delta_{23}} = \frac{\delta_{25}}{\delta_{15}}\frac{\delta_{13}}{\delta_{23}} \qquad \frac{\Delta_{35}}{\Delta_{15}}\frac{\Delta_{12}}{\Delta_{23}} = \frac{\delta_{35}}{\delta_{15}}\frac{\delta_{12}}{\delta_{23}} \qquad \frac{\Delta_{45}}{\Delta_{15}}\frac{\Delta_{12}}{\Delta_{24}} = \frac{\delta_{45}}{\delta_{15}}\frac{\delta_{12}}{\delta_{24}}. \tag{29}$$

Equations (29) are understood as a system of linear equations for $x_1^{(5)}$, $x_2^{(5)}$ and $x_3^{(5)}$. This system has a solution when its determinant $D(t)$ is different from zero. By direct calculation for $t = 0$, we obtain

$$\begin{aligned} D(0) = (\delta_{15} - \delta_{25})(\delta_{13} - \delta_{14} + \delta_{34}) - (\delta_{15} - \delta_{35})(\delta_{12} - \delta_{14} + \delta_{24}) \\ + (\delta_{15} - \delta_{45})(\delta_{12} - \delta_{13} + \delta_{23}). \end{aligned} \tag{30}$$

The terms in the second parentheses do not depend on $u_3^{(i)}$ and are not identical zero. Because the terms in the first parentheses depend on $u_3^{(i)} - u_3^{(k)}$, this determinant is not identical zero for any possible initial conditions $\boldsymbol{u}_i$ and $\boldsymbol{U}_i$. As the determinant $D(t)$ is given by the solutions of system (17) which are continuous and $D(0) \neq 0$, there is any neighbourhood of $t = 0$ in which the determinant $D(t) \neq 0$. We see that, in this neighbourhood, we can determine, from the system of equations (29), the solutions $x_1^{(5)}$, $x_2^{(5)}$ and $x_3^{(5)}$ of the system of differential equations (17) by using the particular solution $\boldsymbol{x}_i$, $\boldsymbol{Y}_i$ for $i = 1, \ldots, 4$. In other words, formulae (27) give the implicit form of the nonlinear superposition formulae for the system of differential equations (17) which is connected with the action of the Lie group $SO(3, 2)$ on space M.

Comments. Equations (28) imply that the four solutions are not fully independent. For example, if we know three solutions $\boldsymbol{x}_i$, $\boldsymbol{Y}_i$, $i = 1, 2, 3$, and from the fourth we know $x_3^{(4)}$, we can obtain $x_1^{(4)}$ and $x_2^{(4)}$ from (28). This is a consequence of the fact that the reconstruction of the action of the group requires only ten independent functions.

5. Conclusions

The main results of this paper are the following:

(1) We constructed systems of first-order ordinary differential equations that arise from the infinitesimal action of the local Lie group $SO(n+1, n)$ on the factor space $M = SO(n+1, n)/P$, where P is one of the maximal parabolic subgroups of $SO(n+1, n)$. These systems allow a superposition formula.

(2) We found all sets of invariants for these systems which are expressed in terms of the solutions. These are invariants of the matrices Ω_{ik}. The matrices Ω_{ik} play, in our case, a similar role to the matrix anharmonic ratio for projective-matrix Riccati equations in [4].

(3) In the case of $SO(3, 2)$, we proved that, from these invariants, it is possible to find the general solution of our system on the basis of four particular solutions. Therefore, in this case, this set of invariants gives an implicit nonlinear superposition formula. It is necessary to note that, even though the local groups $SO(3, 2)$ and $SP(4, \boldsymbol{R})$ are isomorphic, our system of differential equations differs from the one studied in [4] for the Lie group $SP(4, \boldsymbol{R})$, because we use another maximal parabolic subgroup for constructing the space M.

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